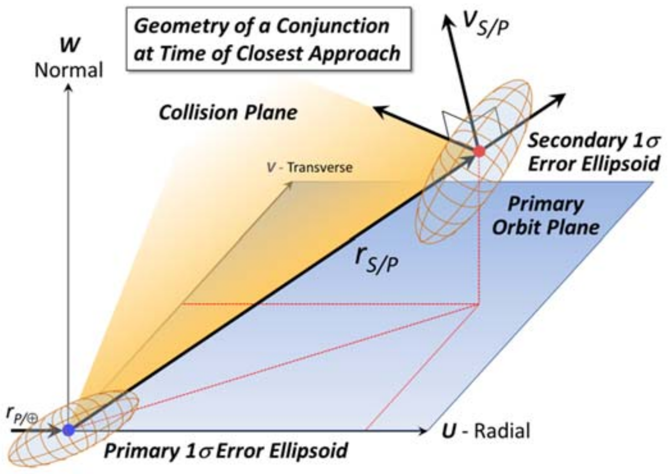

Probability of Collision in the Joint Space Operations Center

For a conjunction[1] between two objects in earth orbit, the Joint Space Operations Center (JSpOC) can compute and report a value commonly referred to as probability of collision (Pc[2]). This value can be used by a satellite owner/operator (O/O) as a warning and may motivate further study of the encounter on their part[3].

The data at the time of closest approach (TCA) required to compute Pc includes:

- The size of the primary object
- The size of the secondary object
- The inertial position and velocity vectors of the primary object
- The inertial position and velocity vectors of the secondary object
- The 3-dimensional position covariance of the primary object
- The 3-dimensional position covariance of the secondary object

If available and reliable these data items are provided in the Conjunction Data Message (CDM).

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[1] Also referred to as a “close approach”

[2] Also PoC, POC

[3] When an assumption is made in the calculation of Pc, it’s made on the side of overestimating the Pc, in other words, being conservative. In this way O/O’s are made aware of all potential encounters. The first place where this conservatism appears is in what is actually calculated. What is computed is not probability of collision but rather the probability that two objects are less than a specified distance apart at their time of closest approach (TCA). This is due to several factors, the JSpOC not knowing the exact size, shape and orientation of both objects at TCA being one of the major ones. So the approach is to determine the longest distance the two satellite’s centers of mass can be apart and still have the two satellites touch. This defines the “specified distance apart” that is used in the Pc calculation. Note that if the two satellites are not spheres, then a simple change in orientation means that they may not touch and no collision would occur. So the Pc value calculated always has this factor of conservatism built in.

Object Size

The Pc calculation requires an estimate of the sizes of the primary and secondary objects. Because of a dimensionality reduction of the problem (to be described subsequently), the combined sizes of the two objects ultimately will be represented as an area in a projected plane—the "conjunction plane." The typical procedure is to circumscribe the primary and secondary objects each by a sphere, add the two sphere radii to generate a supervening sphere that can contain both circumscribing spheres, and project this supervening sphere into the conjunction plane as a circle.

The size of the objects[4] is called AREA_PC in the CDM and given in square meters. The CDM Blue Book[5] states:

> *AREA PC: The actual area of the object (m**2). The area could be known by the owner/operator of the satellite or defined by using a Radar Cross Section (RCS) as in the case of debris. If the value of the area is unknown or not available, "0.0" may be displayed. This parameter can be useful for calculation collision probability.*

SuperCOMBO[6] is used to identify conjunctions in the JSpOC. It has three modes: standoff radius, ellipsoid, and covariance. Pc is computed in covariance mode. It requires that the sizes of both objects be input by the operator. Size is specified (in meters) by the Radius of Exclusion Volume; as stated above, it represents the minimum radius of a sphere that can contain the object[7] [8]. SuperCOMBO can also assign specific values or use an RCS catalog to establish size.

Caution is advised if RCS is used to establish size. For many reasons RCS may not represent the physical size of an object, and certainly not its shape or orientation at TCA.

Position and Velocity Vectors

The position and velocity vectors[9] of the primary and secondary objects at TCA are given in the CDM in the fields X, Y, Z, X_DOT, Y_DOT, and Z_DOT. Position units are km and velocity units are km/sec. They are normally referenced to the International Terrestrial Reference Frame (ITRF), but other coordinate frames may be specified in accordance with the CDM Blue Book. The velocity vectors are not explicitly used in the computation of Pc. However, they are required in order to establish the point and time of

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[4] The primary and secondary objects are referred to as OBJECT1 and OBJECT2 in the CDM.

[5] Conjunction Data Message Recommended Standard CCSDS 508.0-B-1 BLUE BOOK, June 2013

[6] Super Computation of Miss Between Orbits. SuperCOMBO is an application program in the Astrodynamics Support Workstation (ASW). It is based on the Air Force Space Command (AFSPC) Astrodynamics Standards program COMBO.

[7] If an O/O reports object size to the JSpOC in m**2 then that should be the area of a circle that completely "covers" the object when viewed from any angle. The JSpOC would then compute the Radius of Exclusion Volume from this area as (O/O-reported-size divided by π)**0.5.

[8] Pre-assigned default values for payloads and platforms (5 meters), rocket bodies and unknown objects (3 meters) and debris (1 meter) were determined through a study of sizes of objects in the space object catalog and are normally used. Plans are being developed to allow O/O's to provide a more accurate value of this parameter for use in CA screening in the future.

[9] Together, often referred to as the state vector

closest approach, to perform needed coordinate transformations/rotations to prepare the data properly for the Pc calculation, and to determine encounter geometry at TCA.

In the JSpOC the position and velocity of the objects at TCA are normally computed by interpolating between ephemeris points[10]. The ephemeris points are obtained by propagating forward in time the object's special perturbations (SP) state vector at epoch[11]. The state vector at epoch is computed using an orbit determination method based on minimum variance differential correction (DC).

In some cases ephemeris points of an object are provided to the JSpOC by the satellite owner/operator[12]. If so, these are used in lieu of, or in addition to, internal ephemeris.

While not explicitly input to compute Pc, the position and velocity of the secondary relative to the primary at TCA are included in the CDM. They are given in Radial, Transverse, and Normal (RTN) coordinates of the primary object[13]. Also provided are overall MISS_DISTANCE and RELATIVE_SPEED. Relative position and velocity offer a more intuitive view of the conjunction[14] and tell the user if the secondary is moving fast or slow relative to the primary at TCA.

Covariance and Error Ellipsoids

The covariance matrix, provided separately for the primary and secondary object (as described in Annex C of CCSDS 508.0-B-1), is routinely furnished in 6 x 6 form, even though only the position portion is needed for the Pc calculation as described in this memorandum. A covariance matrix characterizes the uncertainty in a satellite state vector, similar to the way that variance, the square of the standard deviation, is a metric for assessing the spread of student test scores about the mean. An example of a position covariance, using the notation of the CCSDS standard, is given as the symmetric matrix below. The diagonal elements represent the variance in each of the components (R, T, and N), and the off-diagonal terms give the covariance between the two named components (the product of the two components' standard deviations and their correlation coefficient):

$$Cov = \begin{pmatrix} CR\_R & CT\_R & CN\_R \\ CT\_R & CT\_T & CN\_T \\ CN\_R & CN\_T & CN\_N \end{pmatrix}$$

The covariance matrix for each object is referenced to its own RTN coordinate frame. For each object matrix components are computed using 5-point Lagrange interpolation of the covariance in the ephemeris file produced by the JSpOC[15].

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[10] An ephemeris is a time ordered listing of an object's position and velocity vectors (that may include covariance).

[11] Or general perturbations (GP) element set (ELSET) in the absence of an SP state vector

[12] In J2000 mean equator mean equinox coordinates

[13] Also called the Radial-Intrack-Crosstrack (RIC) coordinate frame or the UVW system

[14] For example, at TCA the secondary relative to the primary is at "eleven o'clock high."

[15] Covariance at TCA quantifies uncertainty in the state vector at TCA. Covariance at epoch quantifies uncertainty at epoch. Covariance at epoch in essence maps the observed-minus-computed residuals in the observations used in the DC to the uncertainty in the state vector produced by the DC.

An owner/operator may also send to the JSpOC an ephemeris that includes covariance[16]. In these cases the covariance values in the CDM are interpolated from the ephemeris (with covariance) provided by the owner/operator.

The one-standard-deviation[17] three dimensional (3D) error ellipsoid may be used to visualize the "size" of the position uncertainty at TCA. Diagonalization of the 3X3 covariance matrix yields the size, shape, and orientation of the error ellipsoid[18]. Often, because of drag, the largest uncertainty will be in the direction of the satellite's motion relative to the atmosphere[19].

Assumptions

When computing Pc in the JSpOC several assumptions are made. These include:

- Object sizes are known or can otherwise be assigned an upper bound.
- The conjunction is "hyperkinetic, meaning that the conjunction duration is very short; this allows the additional simplifying assumption that the relative motion between objects is rectilinear throughout the encounter.
- Gaussian theory and statistics apply.
- Covariance for both objects is known and constant throughout the encounter.
- Primary and secondary errors are independent allowing "combined" covariance to be the simple sum of the individual covariances (in a common frame), resulting in the "joint" covariance.
- The covariance is not "too large."
- The covariance is not "too small[20]."

Technical papers have been written to address the computation and/or validity of Pc when one or more of these assumptions come into question. If at TCA (1) covariance is too small, (2) covariance is too large, or (3) the relative speed of the secondary relative to the primary is too small, then Pc is not computed in SuperCOMBO. The values for covariance "too large" and relative speed "too small" are user-settable.

Another concern is covariance realism: does the covariance used in the computation of Pc truly reflect the uncertainty in the state vectors at TCA? Because quantitative studies have shown that covariance is often underestimated, empirical techniques have been devised to scale or otherwise inflate the covariance – make it larger – to compensate for the underestimation.

Computation

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[16] Owner/operators provide ephemeris with covariance in either of two formats: (1) the Generic On-Orbit (GOO) ephemeris data format or (2) the Modified ITC ephemeris data format. See the *Ephemeris Guide for Operators.*

[17] 1 sigma or 1σ

[18] The square root of each of the eigenvalues of the covariance matrix gives the sizes of the ellipsoid axes. The eigenvectors provide the orientation of the ellipsoid axes relative to the object's RTN coordinate frame.

[19] A very rough estimate of the sizes of the three axes of the error ellipsoid can be obtained by taking the square root of each of the diagonal elements CR_R, CT_T, and CN_N.

[20] This test is needed to trap zero covariance.

Computation of Pc takes place in the collision plane[21]. This is the plane perpendicular to the relative velocity vector at TCA. This reduces the mathematics from 3D to 2D, and the kinematics from dynamic to static. In 2D the equation used to compute Pc is:

$$PoC = \frac{1}{2\pi \,|\, Det(C)\,|^{1/2}} \iint\limits_{x^2+y^2 \le d^2} \exp\left(-\frac{1}{2} (\mathbf{r} - \mathbf{r}_{S/P})^T C^{-1} (\mathbf{r} - \mathbf{r}_{S/P}) \right) dxdy$$

In this double integral equation:

- $\pi = 3.141592653589793$.
- C is the 2X2 projection of the combined 3X3 covariance at TCA onto the collision plane[22].
- $Det(C)$ is the determinant of C.
- d is the sum of the two object sizes[23].
- “exp” is the exponential function, i.e., e to the power in parentheses with e = 2.718281828459045.
- $\mathbf{r} = (x, y)^T$ is any point in the collision plane such that $x^2 + y^2 \le d^2$.
- $\mathbf{r}_{S/P} = (r_{S/P}, 0)^T$ is the position of the secondary relative to the primary along the x-axis in the collision plane.
- C^{-1} is the inverse of C.

This equation integrates the 2D Gaussian probability density function (centered on the secondary object) over the circle of radius d (centered on the primary object[24]).

Methodologies, coordinate transformations, assumptions, combining Gaussian distributions, limitations, and techniques used to facilitate computation of Pc are discussed at length in technical papers listed in the bibliography[25].

Notes

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[21] Also called the encounter plane

[22] The combined covariance is the sum of the covariance matrices corresponding to each object (both referenced to the same coordinate frame). The applicability of Gaussian statistics allows for combining two covariance matrices into one using term-by-term summation.

[23] The sum of the radii of exclusion volumes for the two objects

[24] With the primary object located at the origin of a 2D coordinate frame in the collision plane

[25] The computation of Pc is a dynamic 3D problem: the two objects are moving in 3-dimensions. However, it can be simplified by reducing the dynamic 3D problem to a static 2D problem. This is accomplished by performing calculations in the 2D collision plane. Consider the two error ellipsoids of the primary and the secondary objects at the instant of TCA as they “punch holes” in the collision plane that’s perpendicular to the relative velocity vector. The problem of computing Pc immediately collapses to a static 2D problem in the collision plane. The ellipsoids become ellipses in the collision plane representing the 1σ “contour lines” of the two-dimensional Gaussian probability density functions (pdf) of the objects. One can think of the 2D Gaussian (“bell-shaped”) probability density functions for the primary and the secondary as being “little hills” sitting over their respective satellite positions at TCA in the collision plane. In effect, the ellipse for one object is but one of many nested “Russian Doll” contour lines going out to infinity in the 2D collision plane for the Gaussian pdf, just as in 3D the error ellipsoid is but one of an infinite number of nested Russian dolls. These two “little hills” combine into one hill – one pdf – centered over the secondary for the actual computation of the Pc.

The JSpOC uses error functions (ERF) for computing the double integration in the Pc equation. In addition the JSpOC performs integration over a square circumscribing the circle of radius d. This square is aligned with the axes of the combined 2D probability density function in the collision plane. This simplifies the computation of Pc but gives a very slightly larger – but safe – value.

For these and other reasons agencies may compute slightly different values for Pc for the same conjunction. If the difference is large however further investigation is warranted[26].

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[26] In 2000 AFSPC/A9 numerically validated the computation of Pc in SuperCOMBO.

Bibliography

Several excellent references for understanding the computation of Pc are listed at the Space Assigned Number Authority (SANA) registry at http://sanaregistry.org/r/cdm_cpm/cdm_cpm.html.

In addition the following references are valuable because in the 1990s Doctor Foster matured much of the theory and methodology used by NASA to compute Pc for Space Shuttle and International Space Station (ISS) safety of flight analysis. A version of the NASA algorithm for computing Pc was integrated into the ASW program SuperCOMBO in 2000. Certain theoretical issues were modified and expanded by F.C. Chan in his important monograph *Spacecraft Collision Probability* (El Segundo, CA: Aerospace Press, 2008).

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